New View on an Anisotropic Medium and Its Application to Transformation from Anisotropic to Isotropic Problems

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Abstract—The metric factor is defined as

$$m(\epsilon_x^*,\epsilon_y^*,\theta_x)\equiv\sqrt{\cos^2\theta_x/\epsilon_x^*+\sin^2\theta_x/\epsilon_y^*}$$

in the radial direction, with the angle θ_x from the x axis being one of the principal axes in an anisotropic dielectric medium filling the two-dimensional space. The normalized metric factor is defined as

$$n(\epsilon_x^*,\epsilon_y^*,\theta_x,\beta)\equiv m(\epsilon_x^*,\epsilon_y^*,\theta_x)/m(\epsilon_x^*,\epsilon_y^*,\beta)$$

in the form normalized by the metric factor in the direction with the angle β from the x axis. The effective path length $d'_{P_1P_2}$ between the points P_1 and P_2 is defined as

$$d'_{P_1P_2}=n(\epsilon_x^*,\epsilon_y^*,\theta_x,\beta)d_{P_1P_2}$$

where $d_{P_1P_2}$ is the actual path length of the straight line P_1P_2 with the angle θ_x from the x axis. We propose the minimum principle of the effective path length for electric flux in the region with multilayered anisotropic media. It is applied to solving the electrostatic problem with two anisotropic media whose principal axes are different. We show by using the normalized metric factor that the anisotropic problem can be transformed into the isotropic problem.

I. Introduction

THE boundary value problem for microstrip on anisotropic substrates has received considerable attention [1]–[4], [6]–[8], [16]–[19].[1] The calculation of the parameters of the microstrip transmission line or the electrooptic light modulator line was treated in [1], [4]–[8]. One of the authors extended the method by Silvester [9] for deriving the Green's function for the microstrip line with an anisotropic substrate [1] and proposed, by using this Green's function, a method which calculates with a high accuracy the line capacitance of the microstrip line for anisotropic substrates. This method also calculates accurately the effective filling fraction [10], [11] for the isotropic substrate. The Green's function technique which is useful in solving such a problem was treated in [2]. The authors obtained the exact resistance of two-dimensional anisotropic compound region with antisymmetry [3]. Yamashita *et al.* [6] and Szentkuti [18], [19] treated the method transforming the anisotropic problem into the isotropic problem by using the affine transformation. One of the authors proposed mapping between two-dimensional anisotropic regions with different permittivity tensors and showed two examples which were able to be transformed the single anisotropic into isotropic regions [12].

In this paper, the metric factor and the normalized metric factor of an anisotropic medium are defined to show what properties the electric flux considered holds. The minimum principle of effective path length for such electric flux is expressed in the form of integration by using the normalized metric factor. Applying this principle, the electrostatic problem with two anisotropic media, whose principal axes are different from each other, is solved. Also, we show that the electrostatic problem with multilayered anisotropic media can be transformed into that with multilayered isotropic media by using the normalized metric factor. It is valuable from the viewpoint of the boundary value problem to solve the anisotropic problem as it is. Also, it is useful to obtain the method being able to transform the anisotropic problem into the isotropic problem because we can use the many available methods for the analysis of the isotropic problem. We represent from the fact of this transformation that the line capacitance per unit length for the microstrip line with anisotropic dielectric substrate is equal to that for the microstrip line with a corresponding isotropic dielectric substrate. Therefore, we can easily calculate the former line capacitance by using the accurately approximate formula of effective filling fraction for the case of isotropic dielectric substrate. This result will be shown in the other paper [15].

II. Metric Factor and Normalized Metric Factor

Now, consider the anisotropic medium of the following permittivity tensor filling the two-dimensional space:

$$\bar{\epsilon}=\begin{bmatrix}\epsilon_x^* & 0\\ 0 & \epsilon_y^*\end{bmatrix}\epsilon_0 \qquad (1)$$

where ϵ_x^* and ϵ_y^* are the relative dielectric constants and ϵ_0 is the permittivity of vacuum. In such a medium, the

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[1]The authors received, on August 9, 1979, the copy of this article from Dr. B. T. Szentkuti by requesting it. He derived the transformation from anisotropic to isotropic problems by the affine transformation based on the Fourier transformation. He treated the case of two-dielectric region in which one is isotropy and the other is anisotropy, and suggested that the isotropy transformation will be able to be derived for the more general cases. We can find that his results (A3.16a)–(A3.16c) are identical to our results (45), (46), (48) after the next correction of (A3.8),

$$\alpha=\sqrt{(\kappa_{xx}/\kappa_{yy})-(\kappa_{xy}^2/\kappa_{yy})}\ \to\ \sqrt{(\kappa_{xx}/\kappa_{yy})-(\kappa_{xy}^2/\kappa_{yy}^2)}\ .$$

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electric potential ϕ at an arbitrary point (x,y) for the line charge q_0 at the source point (x_0,y_0) is the solution of the two-dimensional inhomogeneous partial differential equation:

$$\epsilon_x^* \frac{\partial^2 \phi}{\partial x^2} + \epsilon_y^* \frac{\partial^2 \phi}{\partial y^2} = -\frac{q_0 \delta(x-x_0)\delta(y-y_0)}{\epsilon_0}. \quad (2)$$

Applying the coordinate transformation $(u = x/\sqrt{\epsilon_x^*}$, $v = y/\sqrt{\epsilon_y^*}$) and the property of the delta function ($\delta(ax) = \delta(x)/|a|$) to (2), we get

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = -\frac{q_0}{\epsilon_0 \sqrt{\epsilon_x^* \epsilon_y^*}} \delta(u-u_0)\delta(v-v_0). \quad (3)$$

Therefore, we can obtain the solution of (2) by applying the coordinate transformation inversely to the solution of (3), that is,

$$\phi = \frac{q_0}{2\pi\epsilon_0\sqrt{\epsilon_x^*\epsilon_y^*}} \ln \frac{c}{\sqrt{\frac{1}{\epsilon_x^*}(x-x_0)^2 + \frac{1}{\epsilon_y^*}(y-y_0)^2}} \quad (4)$$

where c is an arbitrary constant.

We see from (4) that the line charge emits electric flux in the radial direction and the electric flux ψ per unit angle in the radial direction with the angle θ_x from the x axis, which is one of principal axes, is expressed as follows:

$$\psi(q_0, \epsilon_x^*, \epsilon_y^*, \theta_x) \equiv \frac{q_0}{2\pi} \frac{\sqrt{\epsilon_x^* \epsilon_y^*}}{\epsilon_x^* \sin^2\theta_x + \epsilon_y^* \cos^2\theta_x}. \quad (5)$$

Now, we define the metric factor m in the radial direction with the angle θ_x from the x axis in such an anisotropic medium as follows:

$$m(\epsilon_x^*, \epsilon_y^*, \theta_x) \equiv \sqrt{\frac{1}{\epsilon_x^*}\cos^2\theta_x + \frac{1}{\epsilon_y^*}\sin^2\theta_x}. \quad (6)$$

Therefore, we can say that this metric factor is the factor to show the degree of anisotropy of the anisotropic medium.

Also, we define the normalized metric factor n as the metric factor normalized by the metric factor in the direction with the angle β from the x axis, that is,

$$n(\epsilon_x^*, \epsilon_y^*, \theta_x, \beta) \equiv m(\epsilon_x^*, \epsilon_y^*, \theta_x)/m(\epsilon_x^*, \epsilon_y^*, \beta). \quad (7)$$

We define the effective path length $d'_{P_1P_2}$ between the point $P_1(x_1,y_1)$ and the point $P_2(x_2,y_2)$ as follows:

$$d'_{P_1P_2} = n(\epsilon_x^*, \epsilon_y^*, \theta_x, \beta) d_{P_1P_2} \quad (8)$$

where

$$d_{P_1P_2} = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \quad (=\text{actual path length}) \quad (9)$$

and θ_x is the angle between the direction of the line P_1P_2 and the x axis. Using this effective path length, we can express the contribution ϕ to the electric potential at the point $P_2(x_2,y_2)$ due to only the line charge q_0 at the point $P_1(x_1,y_1)$ as follows:

$$\phi = q_0 \ln(1/d'_{P_1P_2}) / \left(2\pi\epsilon_0\sqrt{\epsilon_x^*\epsilon_y^*}\right). \quad (10)$$

The β in (7) is defined as the angle between the interface of different media and the x axis being one of principal axes of each medium. On the other hand, the β for the case of single medium is arbitrary because there is no interface and the two-dimensional electric potential ϕ in such a medium has an arbitrary constant c (see (4)), not determined unless the value of ϕ at the proper point is given. Therefore, we can say from (6)–(10) that to define the normalized metric factor n as (7) means to let the electric potential ϕ at P_2 due to the line charge q_0 at P_1 be zero for the case of single medium, where the direction of straight line P_1P_2 is parallel to the imaginary interface with the proper β from the x axis and $d_{P_1P_2} = 1$.

The typical final-electric flux line, the final solution associated with the line charge q_0 in the region with the two anisotropic media, is shown in Fig. 1(a). We consider here the case which the permittivity tensors of two anisotropic media are expressed in the form of (1), respectively. We can resolve this final-electric flux into the refracted and reflected flux [1]. For example, the final-electric flux at the field point D can be obtained by using the electric flux emitted from the source line charge q_0 at the source point $Q_0(x_0,y_0)$, and the reflected flux which is the electric flux emitted from the image line charge $Kq_0(K = (\sqrt{\epsilon_{2x}^*\epsilon_{2y}^*} - \sqrt{\epsilon_{1x}^*\epsilon_{1y}^*})/(\sqrt{\epsilon_{2x}^*\epsilon_{2y}^*} + \sqrt{\epsilon_{1x}^*\epsilon_{1y}^*}))$ at the image point $Q_2(-x_0,y_0)$ in Fig. 1(b). Also, the final-electric flux at the field point C is identical to the refracted flux which is the electric flux emitted from the image line charge $(1-K)q_0$ at the image point $Q_1(\alpha x_0,y_0)(\alpha = \sqrt{\epsilon_{2y}^*/\epsilon_{2x}^*}/\sqrt{\epsilon_{1y}^*/\epsilon_{1x}^*})$ in Fig. 1(b).

In this paper, we do not pay attention to the final-electric flux in Fig. 1(a), but the electric flux, the refracted and reflected flux such as shown in Fig. 1(b), into which the final-electric flux is resolved. This electric flux paid attention is one emitted from a line charge in the fullspace filled with a single anisotropic dielectric medium and goes from a line charge in the radial direction.

Now, we consider the case of anisotropic media in Fig. 1(b). We can obtain that the effective path lengths of the paths, Q_0B, Q_1B, and Q_2B, have the following relations:

$$d'_{Q_0B} = d'_{Q_1B} = d'_{Q_2B} \quad (11)$$

where

$$d'_{Q_0B} = n(\epsilon_{2x}^*, \epsilon_{2y}^*, \theta_2, \pi/2) d_{Q_0B} \quad (12)$$

$$d'_{Q_1B} = n(\epsilon_{1x}^*, \epsilon_{1y}^*, \theta_1, \pi/2) d_{Q_1B} \quad (13)$$

$$d'_{Q_2B} = n(\epsilon_{2x}^*, \epsilon_{2y}^*, \theta_2, \pi/2) d_{Q_2B}. \quad (14)$$

Therefore, the effective path lengths $d'_{Q_0Q \text{ via } B}$ and $d'_{Q_0C \text{ via } B}$ are expressed as follows:

$$d'_{Q_0A \text{ via } B} = n(\epsilon_{2x}^*, \epsilon_{2y}^*, \theta_2, \pi/2) d_{Q_0B} + n(\epsilon_{2x}^*, \epsilon_{2y}^*, \theta_2, \pi/2) d_{BA}$$
$$= n(\epsilon_{2x}^*, \epsilon_{2y}^*, \theta_2, \pi/2) d_{Q_2A} \quad (15)$$

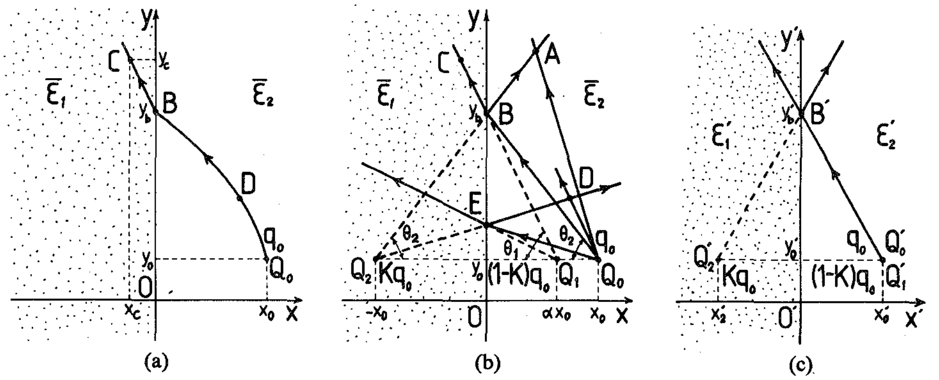


Fig. 1. Typical electric flux lines in the region with two anisotropic media (the actual path length region G) and in the region with two isotropic media (the effective path length region G'). (a) Typical final-electric flux line in G. (b) Typical electric flux lines in G. (c) Typical electric flux lines in G'. $y_b=y_c\alpha x_0/(\alpha x_0-x_c)$, $\alpha=\sqrt{\epsilon^*_{2y}/\epsilon^*_{2x}}\,/\sqrt{\epsilon^*_{1y}/\epsilon^*_{1x}}$, $\epsilon'_1=\epsilon^*_1\epsilon_0$, $\epsilon'_2=\epsilon^*_2\epsilon_0$, $\epsilon^*_1=\sqrt{\epsilon^*_{1x}\epsilon^*_{1y}}$, $\epsilon^*_2=\sqrt{\epsilon^*_{2x}\epsilon^*_{2y}}$, $K=(\epsilon^*_2-\epsilon^*_1)/(\epsilon^*_2+\epsilon^*_1)$, $x'_0=x_0\sqrt{\epsilon^*_{2y}/\epsilon^*_{2x}}$, $x'_2=-x'_0$, $y'_0=y_0$, $y'_b=y_b$.

$$d'_{Q_0C\text{ via }B}=n(\epsilon^*_{2x},\epsilon^*_{2y},\theta_2,\pi/2)d_{Q_0B}+n(\epsilon^*_{1x},\epsilon^*_{1y},\theta_1,\pi/2)d_{BC}$$
$$=n(\epsilon^*_{1x},\epsilon^*_{1y},\theta_1,\pi/2)d_{Q_1C}. \quad (16)$$

We easily find from Fig. 1(b) that the path Q_0BA is one with the shortest path length, but the path Q_0BC is not so. However, we can find from (15) and (16) that the paths Q_0BA and Q_0BC are the paths with the shortest effective path lengths.

If the two media are isotropic in Fig. 1(b), $\epsilon^*_{ix}=\epsilon^*_{iy}=\epsilon^*_i(i=1,2)$, the image point Q_1 becomes the position of the source point Q_0. Then, we can easily find that the paths Q_0BC and Q_0BA are the paths with shortest path length among the paths of the electric flux travelling via the point on the interface from Q_0 to C and from Q_0 to A, respectively. Also, we can find by letting $\epsilon^*_{ix}=\epsilon^*_{iy}=\epsilon^*_i$ in (15) and (16) and using $n(\epsilon^*,\epsilon^*,\theta_x,\beta)\equiv 1$ for the isotropic medium that the paths, Q_0BA and Q_0BC, are the paths with the shortest effective path lengths.

From these results, we can say that the electric flux emitted from the source point travels as such that its effective path length becomes shortest. Therefore, we propose the minimum principle of effective path length for electric flux of an electrostatic problem with anisotropic media which will be able to be solved by using the image-coefficient method [1] as follows:

$$\int_{\text{path}} n(\epsilon^*_x,\epsilon^*_y,\theta_x,\beta)\,ds=\text{minimum}. \quad (17)$$

This principle states that the electric flux emitted from the source line charge chooses a trajectory that minimizes the effective path length. We must take note that this principle is valid for the refracted and reflected flux into which the final-electric flux associated with a single line charge at an arbitrary point in the region with anisotropic media is resolved, but not valid for the final-electric flux. Also, this principle can be applied to the case with an arbitrary number of the media (even whose principal axes are different) if those interfaces are parallel, and to the finite region bounded by Dirichlet, Neumann, or those mixed boundaries whose boundaries are piecewise smooth. We can obtain a trajectory of an electric flux emitted from the source point in Fig. 1(b) by solving the Euler equation for the principle (17). Therefore, we can obtain the positions of the image points Q_1 and Q_2 by using (11). Then, we can find that these image points are fixed if the source point is given.

In order to show the validity of this minimum principle, let us solve the problem which has not been solved in the literature, with the interface at $y=b_1x$, with two anisotropic media whose principal axes are different from each other, and with the line charge q_1 at the source point $P_1(x_1,y_1)$ in the medium $\bar{\epsilon}_1$, as shown in Fig. 2. It is difficult to solve this problem by using only the image-coefficient method in [1]. We express below the permittivity tensor $\bar{\epsilon}_1$ of one anisotropic medium in the x-y coordinates:

$$\bar{\epsilon}_1=\begin{bmatrix}\epsilon^*_{1x} & 0\\ 0 & \epsilon^*_{1y}\end{bmatrix}\epsilon_0 \quad (18)$$

and the permittivity tensor $\bar{\epsilon}_2$ of other anisotropic medium in the ξ-η coordinates:

$$\bar{\epsilon}_2=\begin{bmatrix}\epsilon^*_{2\xi} & 0\\ 0 & \epsilon^*_{2\eta}\end{bmatrix}\epsilon_0. \quad (19)$$

The ξ-η coordinates is obtained by rotating the x-y coordinates with the angle γ as shown in Fig. 2, that is,

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}=\begin{bmatrix}\cos\gamma & \sin\\ -\sin\gamma & \cos\end{bmatrix}\begin{bmatrix}\gamma\ x\\ \gamma\ y\end{bmatrix}. \quad (20)$$

This problem is the fundamental one to derive the Green's function for the microstrip line with the anisotropic substrate whose principal axes are not parallel and perpendicular to the ground conductor. Solving the Euler equation for the principle (17), we can know the trajectory of an electric flux emitted from the source line charge q_1. Let the points $P_2(x_2,y_2)$ and $P_3(x_3,y_3)$ be the image point of the refracted flux P_iP_5 and that of the reflected flux P_iP_4, for the electric flux P_1P_i emitted from the source line charge q_1 at $P_1(x_1,y_1)$, respectively. The points P_2 and P_3

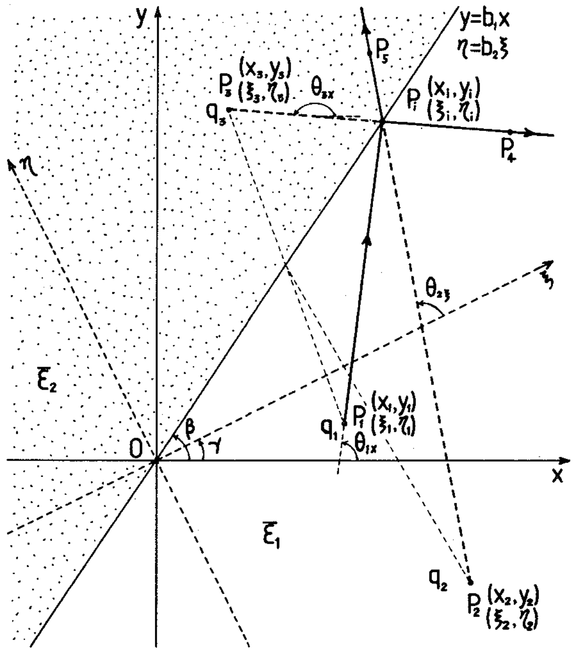


Fig. 2. Refraction and reflection of the electric flux emitted from the source line charge q_1 in the region with two anisotropic media (the actual path length region G).

can be determined, respectively, from the following conditions for the effective path lengths:

$$d'_{P_1P_i}=d'_{P_2P_i}=d'_{P_3P_i} \tag{21}$$

where

$$d'_{P_1P_i}=n_1(\epsilon^*_{1x},\epsilon^*_{1y},\theta_{1x},\beta)d_{P_1P_i} \tag{22}$$

$$d'_{P_3P_i}=n_1(\epsilon^*_{1x},\epsilon^*_{1y},\theta_{3x},\beta)d_{P_3P_i} \tag{23}$$

$$d'_{P_2P_i}=n_2(\epsilon^*_{2\xi},\epsilon^*_{2\eta},\theta_{2\xi},\beta-\gamma)d_{P_2P_i} \tag{24}$$

$$\beta=\tan^{-1}b_1. \tag{25}$$

So that

$$\begin{cases} x_3=\dfrac{y_1}{b_1}+\dfrac{\alpha_1^2-b_1^2}{\alpha_1^2+b_1^2}\left(x_1-\dfrac{y_1}{b_1}\right) \\ y_3=y_1+\dfrac{2b_1\alpha_1^2}{\alpha_1^2+b_1^2}\left(x_1-\dfrac{y_1}{b_1}\right) \end{cases} \tag{26}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}=\begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}\begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} \tag{27}$$

$$\xi_2=\frac{\alpha_1\{\alpha_1\alpha_2(b_1\sin\gamma+\cos\gamma)+b_1(b_1\cos\gamma-\sin\gamma)\}}{\alpha_2(\alpha_1^2+b_1^2)}\cdot\left(x_1-\frac{y_1}{b_1}\right)+(b_1\sin\gamma+\cos\gamma)y_1/b_1 \tag{28}$$

$$\eta_2=\frac{\alpha_1\{\alpha_1(b_1\cos\gamma-\sin\gamma)-\alpha_2b_1(b_1\sin\gamma+\cos\gamma)\}}{\alpha_1^2+b_1^2}\cdot\left(x_1-\frac{y_1}{b_1}\right)+(b_1\cos\gamma-\sin\gamma)y_1/b_1 \tag{29}$$

where

$$\alpha_1=\sqrt{\epsilon^*_{1y}/\epsilon^*_{1x}}\,,\quad \alpha_2=\sqrt{\epsilon^*_{2\eta}/\epsilon^*_{2\xi}}\,. \tag{30}$$

Let the magnitudes of the image charges at P_2 and P_3, here, be denoted by q_2 and q_3, respectively. We can express the electric potential ϕ at the arbitrary point (x,y) from (10) as follows:

$$\phi=\frac{1}{2\pi\epsilon_0\sqrt{\epsilon^*_{1x}\epsilon^*_{1y}}}\cdot\left[q_1\ln\frac{\sqrt{\alpha_1^2+b_1^2}}{\sqrt{1+b_1^2}\sqrt{\alpha_1^2(x-x_1)^2+(y-y_1)^2}}+q_3\ln\frac{\sqrt{\alpha_1^2+b_1^2}}{\sqrt{1+b_1^2}\sqrt{\alpha_1^2(x-x_3)^2+(y-y_3)^2}}\right],$$

$$(x,y)\in\bar{\epsilon}_1 \text{ region} \tag{31}$$

$$\phi=\frac{q_2}{2\pi\epsilon_0\sqrt{\epsilon^*_{2\xi}\epsilon^*_{2\eta}}}\cdot\ln\frac{\sqrt{\alpha_2^2+b_2^2}}{\sqrt{1+b_2^2}\sqrt{\alpha_2^2(\xi-\xi_2)^2+(\eta-\eta_2)^2}},$$

$$(\xi,\eta)\in\bar{\epsilon}_2 \text{ region} \tag{32}$$

where

$$b_2=(b_1\cos\gamma-\sin\gamma)/(b_1\sin\gamma+\cos\gamma). \tag{33}$$

Let us determine q_2 and q_3 as follows:

$$q_2=(1-K)q_1 \tag{34}$$

$$q_3=Kq_1 \tag{35}$$

$$K=\frac{\sqrt{\epsilon^*_{1x}\epsilon^*_{1y}}-\sqrt{\epsilon^*_{2\xi}\epsilon^*_{2\eta}}}{\sqrt{\epsilon^*_{1x}\epsilon^*_{1y}}+\sqrt{\epsilon^*_{2\xi}\epsilon^*_{2\eta}}} \tag{36}$$

where the fraction K is called the image coefficient. The validity of this determination is obtained by investigating the requirements of continuity of the normal final-electric flux density component and the tangential final-electric field component at the interface $y=b_1x$. Therefore, at the interface, the fraction $K\psi(q_1,\epsilon^*_{1x},\epsilon^*_{1y},\theta_{3x})$ of the flux $\psi(q_1,\epsilon^*_{1x},\epsilon^*_{1y},\theta_{1x})$ emitted from the source point P_1 reflects and goes as emitted from the image point P_3, and the remainder $(1-K)\psi(q_1,\epsilon^*_{2\xi},\epsilon^*_{2\eta},\theta_{2\xi})$ refracts and goes as emitted from the image point P_2.

III. Transformation from Anisotropic Problem to Isotropic Problem

We see from (10) that the equipotential line is the curve connecting the points whose effective path length from the source point are equal to each other. This means that the boundary value problem with anisotropic medium can be transformed into that with isotropic medium. Let us call the former region with the actual path-length region G and the latter region with the effective path-length region G'.

For example, we can transform the anisotropic problem shown in Fig. 1(b) into the isotropic problem shown in

Fig. 1(c) as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} n(\epsilon_{ix}^*, \epsilon_{iy}^*, 0, \pi/2) & 0 \\ 0 & n(\epsilon_{ix}^*, \epsilon_{iy}^*, \pi/2, \pi/2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{37}$$

where $i=1$ for the field point $(x,y) \in \bar{\epsilon}_1$ region and the image point $Q_1(x_1,y_1)$, and $i=2$ for the field point $(x,y) \in \bar{\epsilon}_2$ region, the source point $Q_0(x_0,y_0)$, and the image point $Q_2(x_2,y_2)$, respectively.

Also, we can transform the anisotropic problem shown in Fig. 2 into the isotropic problem shown in Fig. 3 as follows:

$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = N(x,y,\beta) \begin{bmatrix} x \\ y \end{bmatrix} \\ \epsilon_1^* = \sqrt{\epsilon_{1x}^* \epsilon_{1y}^*} \end{cases} \tag{38}$$

for the field point $(x,y) \in \bar{\epsilon}_1$ region, the source point $P_1(x_1,y_1)$, and the image point $P_3(x_3,y_3)$, and

$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = R(\delta)R(-\gamma)N(\xi,\eta,\beta-\gamma)R(\gamma) \begin{bmatrix} x \\ y \end{bmatrix} \\ \epsilon_2^* = \sqrt{\epsilon_{2\xi}^* \epsilon_{2\eta}^*} \end{cases} \tag{39}$$

for the field point $(x,y) \in \bar{\epsilon}_2$ region and the image point $P_2(x_2,y_2)$, where

$$N(x,y,\theta) = \begin{bmatrix} n(\epsilon_x^*, \epsilon_y^*, 0, \theta) & 0 \\ 0 & n(\epsilon_x^*, \epsilon_y^*, \pi/2, \theta) \end{bmatrix} \tag{40}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{41}$$

$$\tan\delta = \frac{(\alpha_2 \sin\gamma + b_2 \cos\gamma)\alpha_1 - (\alpha_2 \cos\gamma - b_2 \sin\gamma) b_1}{(\alpha_2 \cos\gamma - b_2 \sin\gamma)\alpha_1 + (\alpha_2 \sin\gamma + b_2 \cos\gamma) b_1}. \tag{42}$$

In (38), $N(x,y,\beta)$ denotes the transformation matrix of the x-y coordinates into the x'-y' coordinates in order to transform the actual path-length region into the effective path-length region. In (39), $R(\gamma)$ denotes the transformation matrix of the x-y coordinates into the ξ-η coordinates. Next, $N(\xi,\eta,\beta-\gamma)$ denotes the transformation matrix of the ξ-η coordinates into the ξ'-η' coordinates in order to transform the actual path-length region into the effective path-length region. Also, $R(\delta)R(-\gamma)$ denotes the transformation matrix of the ξ'-η' coordinates into the x'-y' coordinates in order to match the interface of two media in the $x'-y'$ axes considered as the coordinate axes in the effective path-length region.

Next, we consider the actual path-length region G, shown in Fig. 4(a), with anisotropic media whose interfaces are parallel to the x axis, whose permittivity tensors are

$$\bar{\epsilon}_i = \begin{bmatrix} \epsilon_{x^{(i)}}^* & 0 \\ 0 & \epsilon_{y^{(i)}}^* \end{bmatrix} \epsilon_0, \qquad i=1,2,3,\cdots,n \tag{43}$$

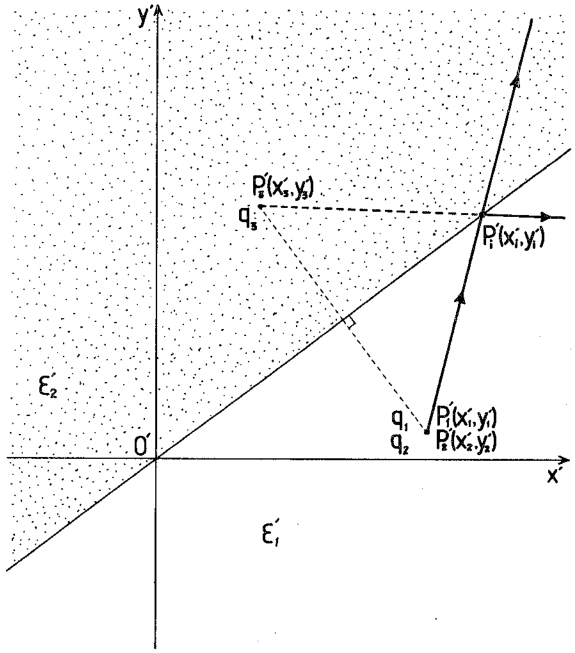


Fig. 3. Refraction and reflection of the electric flux emitted from the source line charge q_1 in the effective path-length region G' transformed from the actual path-length region G in Fig. 2. $\epsilon_1' = \epsilon_1^* \epsilon_0$, $\epsilon_2' = \epsilon_2^* \epsilon_0$, $\epsilon_1^* = \sqrt{\epsilon_{1x}^* \epsilon_{1y}^*}$, $\epsilon_2^* = \sqrt{\epsilon_{2\xi}^* \epsilon_{2\eta}^*}$.

and whose principal axes are such that

$$\begin{bmatrix} x^{(i)} \\ y^{(i)} \end{bmatrix} = R(\gamma_i) \begin{bmatrix} x \\ y \end{bmatrix}, \qquad -\frac{\pi}{2} \leqslant \gamma_i \leqslant \frac{\pi}{2}, \qquad i=1,2,3,\cdots,n. \tag{44}$$

We can transform the region G shown in Fig. 4(a) into the effective path-length region G' shown in Fig. 4(b) as follows, by applying the transformation modified (38) and (39):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T(1) \begin{bmatrix} x \\ y \end{bmatrix}, \qquad (x,y) \in \bar{\epsilon}_1 \text{ region} \tag{45}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T(2) \begin{bmatrix} x \\ y \end{bmatrix}, \qquad (x,y) \in \bar{\epsilon}_2 \text{ region} \tag{46}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T(i) \begin{bmatrix} x \\ y \end{bmatrix} - T(i) \begin{bmatrix} 0 \\ \sum_{k=2}^{i-1} h_k \end{bmatrix} + \sum_{j=2}^{i-1} T(j) \begin{bmatrix} 0 \\ h_j \end{bmatrix},$$

$$(x,y) \in \bar{\epsilon}_i \text{ region}, \qquad i=3,4,5,\cdots,n \tag{47}$$

$$\epsilon_i' = \epsilon_i^* \epsilon_0, \epsilon_i^* = \sqrt{\epsilon_{x^{(i)}}^* \epsilon_{y^{(i)}}^*}, \qquad i=1,2,3,\cdots,n \tag{48}$$

where

$$T(i) = R(\delta_i)R(-\gamma_i)N(x^{(i)}, y^{(i)}, -\gamma_i)R(\gamma_i) \tag{49}$$

$$N(x^{(i)}, y^{(i)}, \theta) = \begin{bmatrix} n(\epsilon_{x^{(i)}}^*, \epsilon_{y^{(i)}}^*, 0, \theta) & 0 \\ 0 & n(\epsilon_{x^{(i)}}^*, \epsilon_{y^{(i)}}^*, \pi/2, \theta) \end{bmatrix} \tag{50}$$

$$\tan\delta_i = (\alpha_i \sin\gamma_i + b_i \cos\gamma_i)/(\alpha_i \cos\gamma_i - b_i \sin\gamma_i) \tag{51}$$

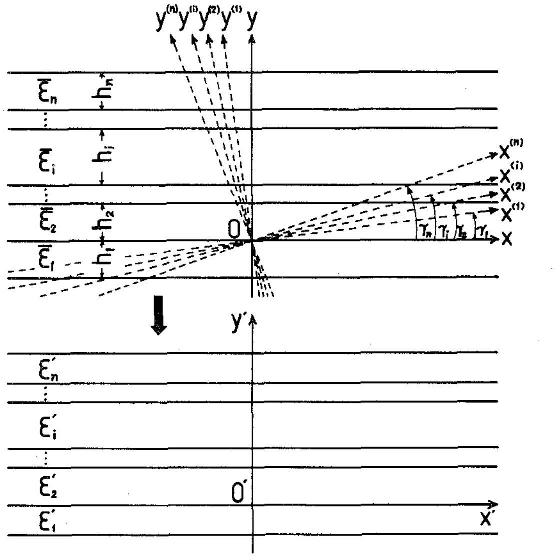


Fig. 4. Transformation from the actual path-length region G into the effective path-length region G'. (a) Actual path-length region G. (b) Effective path-length region G'.

$$b_i = \tan(-\gamma_i) \tag{52}$$

$$\alpha_i = \sqrt{\epsilon^*_{y^{(i)}} / \epsilon^*_{x^{(i)}}}\ . \tag{53}$$

Calculating the matrix $T(i)$ in (49) by using (50)–(53), we can express $T(i)$ with only α_i and γ_i, $i = 1, 2, 3, \cdots, n$, as follows:

$$T(i) = \begin{bmatrix} T_{11}(i) & T_{12}(i) \\ T_{21}(i) & T_{22}(i) \end{bmatrix} \tag{54}$$

where

$$T_{11}(i) = 1, \quad T_{12}(i) = \frac{(\alpha_i^2 - 1)\sin\gamma_i \cos\gamma_i}{(\alpha_i^2 - 1)\cos^2\gamma_i + 1}$$

$$T_{21}(i) = 0, \quad T_{22}(i) = \alpha_i / \{(\alpha_i^2 - 1)\cos^2\gamma_i + 1\}.$$

In the case of $\gamma_i = 0 (i = 1, 2, 3, \cdots, n)$, this transformation is identical to that shown by Yamashita *et al.* [6] and Szentkuti [18]. We know that both the latter transformations are useful for an actual structure as one of the anisotropic substrates' principal axes is parallel to the direction of interface between those substrates in many cases. Szentkuti [19] derived the transformation for the case of $n = 2$, $\alpha_2 = 1$, $\gamma_1 = 0$. His results are identical to our results (see References).

Now we consider the matter obtained from the transformations shown already. We can say that the normalized metric factor is the transforming factor of the path length of the electric flux travelling between two points for being able to treat the anisotropic medium as the isotropic medium. The points on the interface of two media in G, by the transformation (45)–(48), do not slip out of place from each other in G' as the normalized metric factors in two media are unit in the direction parallel to the interface. This means that it is reasonable to define the normalized metric factor as (7). Also, we can find that the effective path lengths and the electric potentials of two corresponding points between G (Fig. 1(b)) and G' (Fig. 1(c)) or G (Fig. 2) and G' (Fig. 3) are mathematically equal to each other, respectively. The electric potential for the case with many line charges can be calculated by the superposition principle. Therefore, we easily see that the line capacitance between the conductors for the region G with anisotropic dielectric media in Fig. 5(a) is equal to that for the region G' in Fig. 5(b) with isotropic dielectric media, which is obtained by using (45)–(48) and (54). Because the total charges on the conductors and the electric potentials of the conductors of two regions are equal to each other for this transformation, respectively. Let the line element of the direction with the angle $\theta_{x^{(i)}}$ from the $x^{(i)}$ axis in the $\bar{\epsilon}_i$ region in G be ds_i, and the total line charge and the charge distribution on it be q_i and σ_i, respectively. Let those in G' be ds_i', q_i', and σ_i', respectively. We obtain the following relation:

$$\sigma_i' = \sigma_i / n(\epsilon^*_{x^{(i)}}, \epsilon^*_{y^{(i)}}, \theta_{x^{(i)}}, -\gamma_i) \tag{55}$$

from $\sigma_i' ds_i' (= q_i') = \sigma_i ds_i (= q_i)$ and $ds_i' = n(\epsilon^*_{x^{(i)}}, \epsilon^*_{y^{(i)}}, \theta_{x^{(i)}}, -\gamma_i) ds_i$. Therefore, we see that the charge distributions on the conductor surfaces parallel to the interfaces in G and G' are identical to each other. We can express $\phi(i)$, $\boldsymbol{E}(i)$, and $\boldsymbol{D}(i)$ in the $\bar{\epsilon}_i$ region in G and $\boldsymbol{E}'(i)$ and $\boldsymbol{D}'(i)$ in the ϵ_i' region in G' by using $\phi'(i)$ in the ϵ_i' region in G' as follows:

$$\phi(i) = \phi'(i) \tag{56}$$

$$\boldsymbol{E}(i) = \boldsymbol{i}E'_{x'}(i) + \boldsymbol{j}\{T_{12}(i)E'_{x'}(i) + T_{22}(i)E'_{y'}(i)\} \tag{57}$$

$$\boldsymbol{D}(i) = \{\boldsymbol{R}(-\gamma_i)\cdot\bar{\epsilon}_i\cdot\boldsymbol{R}(\gamma_i)\}\cdot\boldsymbol{E}(i) \tag{58}$$

$$\boldsymbol{E}'(i) = \boldsymbol{i}E'_{x'}(i) + \boldsymbol{j}E'_{y'}(i) = \boldsymbol{i}\left(-\frac{\partial\phi'(i)}{\partial x'}\right) + \boldsymbol{j}\left(-\frac{\partial\phi'(i)}{\partial y'}\right) \tag{59}$$

$$\boldsymbol{D}'(i) = \epsilon_i'\boldsymbol{E}'(i) \tag{60}$$

where $\boldsymbol{R}(\theta)$ is the tensor expression of $R(\theta)$.

Next, we consider the unsymmetric strip transmission line in G (Fig. 6(a)). The permittivity tensor of its anisotropic dielectric substrate of thickness $2h$ is such that

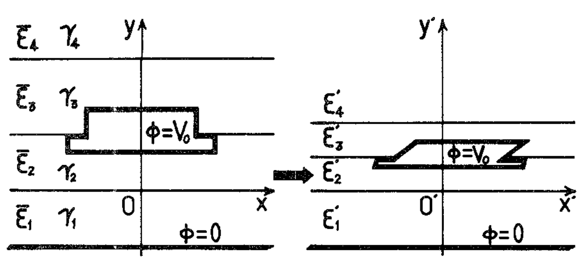
$$\bar{\epsilon} = \begin{bmatrix} \epsilon^*_\xi & 0 \\ 0 & \epsilon^*_\eta \end{bmatrix} \epsilon_0 \tag{61}$$

in the ξ–η coordinates obtained by rotating the x–y coordinates with the angle γ. Its strip width is w, its strip thickness is t, and the slip of the midpoints of two strips is such that

$$d = 2h(\alpha^2 - 1)\sin\gamma\cos\gamma / \{(\alpha^2 - 1)\cos^2\gamma + 1\} \tag{62}$$

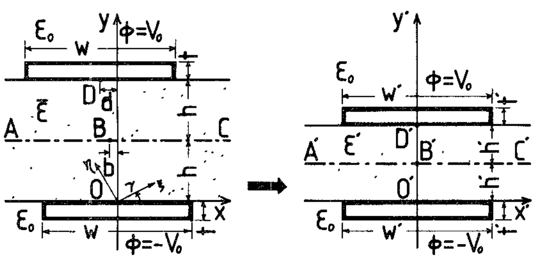
where $\alpha = \sqrt{\epsilon^*_\eta / \epsilon^*_\xi}$. Using (45)–(48) and (54), we can transform the unsymmetric strip transmission line in G into the symmetric strip transmission line, with

$$\epsilon' = \epsilon_0\sqrt{\epsilon^*_\xi \epsilon^*_\eta} \tag{63}$$



(a) (b)

Fig. 5. Example of the transformation from G to G'. (a) Actual path-length region G. (b) Effective path-length region G'. $\alpha_1=\alpha_4=1$, $\alpha_2=2$, $\alpha_3=3$, $\gamma_1=\gamma_2=\pi/6$, $\gamma_3=\pi/4$, $\gamma_4=0$, $\alpha_i=\sqrt{\epsilon_y^{*(i)}/\epsilon_x^{*(i)}}$, $\epsilon_i'=\epsilon_i^*\epsilon_0$, $\epsilon_i^*=\sqrt{\epsilon_x^{*(i)}\epsilon_y^{*(i)}}$, for $i=1,2,3,4$.



(a) (b)

Fig. 6. Transformation from the unsymmetric strip transmission line in G into the symmetric strip transmission line in G'. (a) Unsymmetric strip transmission line in G. (b) Symmetric strip transmission line in G'. $n=3$, $\alpha_1=\alpha_3=1$, $\alpha_2=2$, $h_1=h_3=\infty$, $h_2=2h$, $\gamma_1=\gamma_3=0$, $\gamma_2=\gamma=\pi/6$, $x^{(2)}$-$y^{(2)}$ coordinates $\to\xi$-η coordinates in Fig. 4(a).

$$w'=w \tag{64}$$

$$h'=\alpha h/\{(\alpha^2-1)\cos^2\gamma+1\} \tag{65}$$

$$t'=t \tag{66}$$

in G' (Fig. 6(b)). From Fig. 6, we find that the dot-dash line AC as well as the dot-dash line $A'C'$ are both zero potential lines. Therefore, the microstrip lines with the same anisotropic dielectric substrate higher and lower than the ground conductor are the image lines to the ground conductor, which let the dot-dash line AC be. Also, the line capacitance C/ϵ_0 of each microstrip line in G is equal to that of the corresponding microstrip line in G'. The parameters of a microstrip line with anisotropic substrate can be calculated by using C/ϵ_0 of the case with substrate and C_0/ϵ_0 of the case without substrate. The exact value of C_0/ϵ_0 can be obtained by conformal mapping [13], [14]. The C/ϵ_0 for the microstrip line with anisotropic substrate can be obtained easily from the approximate formula of the effective filling fraction q_w [1], [10], [11] for the case of isotropic substrate. Therefore, it is worthy to show the approximate formula of q_w with a high accuracy. This will be shown in the other paper [15].

We must mention that the transformation (45)–(48) and (54) is valid for the finite region composed of the multi-layered anisotropic media and bounded by the conductor, Neumann, or those mixed boundaries whose boundaries are piecewise smooth, though its example is not shown particularly in this paper.

IV. Conclusion

We have defined the metric factor and the normalized metric factor for an anisotropic medium. Using the normalized metric factor, we have shown the minimum principle of effective path length which shows us a trajectory for an electric flux to travel. In order to show the validity of this principle, we have solved the electrostatic problem with two anisotropic media. Also, it has been shown that the anisotropic problem can be transformed into the isotropic problem by using the normalized metric factor.

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